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SINGULAR PERTURBATIONS OF SPECTRA

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Singular perturbations of spectra*)

by

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ABSTRACT

A mathematical description of free vibrations of a membrane leads to eigenvalue problems for elliptic differential operators containing a small positive parameter ε in the highest order part. The asymptotic behaviour (for $\varepsilon \to +0$) of the eigenvalues is studied in second order problems that reduce to zero-th and first order for $\varepsilon = 0$ and in a fourth order problem that reduces to an elliptic problem of second order. In the case of reduction to zero-th order the density of the eigenvalues on a half-axis grows beyond bound and is proportional to $\varepsilon^{-n/2}$ (in n dimensions). In the case of reduction to first order the relation between the asymptotic behaviour of the spectrum and the critical points of the reduced operator is shown. In the case of reduction to second order an asymptotic series expansion is constructed for every eigenvalue.

KEY WORDS & PHRASES: *Elliptic boundary value problems, singular perturbations, asymptotic expansions of eigenvalues and eigenfunctions, asymptotic density of eigenvalues, Rayleigh's quotient.*

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1. INTRODUCTION

An important aspect in the mechanical theory of plates and shells is the study of vibrations. In a mathematical model for those shells, the relations between deflections, stresses and loads are described by differential equations, the constraints lead to boundary conditions to be imposed, and the free vibrations are represented by eigenvalue problems for those differential equations. A typical equation which describes small deflections W of a clamped membrane of shape Ω, which is stressed uniformly, is

$$(1.1) \qquad \rho \frac{\partial^2 W}{\partial t^2} = N\Delta W, \qquad W_{|\Gamma} = 0, \quad (\Gamma = \text{boundary of } \Omega) ,$$

where ρ is the density per unit area and N the stress. The determination of the free modes $W(x,y,t) = u(x,y)e^{i\omega t}$ naturally leads to the eigenvalue problem

$$(1.2) \qquad \Delta u + \lambda u = 0, \qquad u_{|\Gamma} = 0, \qquad \lambda = \rho\omega^2/N .$$

A more sophisticated model of the same membrane takes into account that the membrane is a shell with finite (small) thickness h and has a flexural rigidity D,

$$D := Eh^3/12(1-\nu^2) ,$$

where E is the elasticity and ν is Poisson's ratio. This leads to the improved model equation, cf. Timoshenko [16, ch. 8],

$$(1.3) \qquad \rho \frac{\partial^2 W}{\partial t^2} = -D\Delta^2 W + N\Delta W, \qquad W_{|\Gamma} = \frac{\partial W}{\partial n}_{|\Gamma} = 0 ,$$

in which D is a small parameter. It looks quite natural that the free modes of (1.3) converge to those of (1.1) if D decreases to zero; we shall prove this in section 5.

We get another type of problem if we consider a membrane on which body forces are exerted and whose tension is weak with respect to those body forces, e.g. a thin metallized membrane in an electromagnetic field. This is described by the model equation, cf. [16, ch. 8],

$$(1.4) \qquad \rho \frac{\partial^2 W}{\partial t^2} = N\Delta W + X\frac{\partial W}{\partial x} + Y\frac{\partial W}{\partial y} , \qquad W_{|\Gamma} = 0 ,$$

where (X,Y) is the body force and may depend on (x,y). In this case the behaviour of the free modes (if present) depends heavily on the field (X,Y). The eigenvalues may disappear at infinity, they may remain discrete or tend to a dense set for $N \to 0$. We shall deal with these problems in sections 3-4.

These mechanical models motivate the study of the following eigenvalue problems on a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary Γ,

$$(1.5) \qquad -\varepsilon\Delta u + p(x,y)u = \lambda u, \qquad u_{|\Gamma} = 0 ,$$

$$(1.6) \qquad -\varepsilon\Delta u + p(x,y)\partial_x u + q(x,y)\partial_y u = \lambda u, \qquad u_{|\Gamma} = 0 ,$$

$$(1.7) \qquad \varepsilon\Delta^2 u - \Delta u = \lambda u, \qquad u_{|\Gamma} = \frac{\partial u}{\partial n}_{|\Gamma} = 0 ,$$

where ε is a small positive parameter, where p and q are smooth real functions on Ω and where λ is the (complex) spectral parameter. We shall study how the eigenvalues of these problems behave as ε decreases to zero.

We shall show that the eigenvalues of problem (1.5) decrease with ε, and that their density (above the minimum of p) increases beyond bound for $\varepsilon \to +0$ and is proportional to $1/\varepsilon$. The eigenvalues of the third problem (1.7) decrease also, but they remain well separated and (as we expect) they converge for $\varepsilon \to +0$ to the eigenvalues of Dirichlet's problem $-\Delta u = \lambda u$, $u_{|\Gamma} = 0$; if Γ is smooth enough we can construct asymptotic series in powers of $\varepsilon^{\frac{1}{2}}$ for the eigenvalues and eigenfunctions. The spectral properties of the second problem (1.6) depend heavily on the characteristics of the first order operator $p\partial_x + q\partial_y$: all eigenvalues may recede to infinity (if Ω does not contain critical points of $dy/dx = p/q$), they may tend to a discrete set or their density may grow beyond bound.

The problems (1.5-6-7) are prototypes of much more general elliptic singularly perturbed boundary value problems in n-dimensional space, for which we can obtain analogous results. We have avoided this greater generality, lest the essential techniques should be obscured by the amount of calculations.

Another motivation for the study of the eigenvalue problem $L_\varepsilon u = \lambda u$, where L_ε stands for an operator defined in (1.5-6 or 7), is the study of the steady state equation $L_\varepsilon u = f$ (+ boundary conditions). It may be dangerous to construct inadvertently a formal approximate solution of $L_\varepsilon u = f$, if zero is the (unknown) limit of an eigenvalue. As an example, we refer to [1], [14] and related papers on the singularly perturbed turning point problem (the one-dimensional analogue of (1.6)), where fallacious and contradictory results were obtained by use of merely formal methods. See also [3].

NOTATIONS

Let Ω be a bounded open set in the plane ($\mathbb{R}^2$) with boundary Γ. It satisfies the *cone condition* if for any point $(x,y) \subset \Omega$ we can place a cone of fixed height h and aperture ω with its top at (x,y) in such a way that the cone is contained inside Ω completely. $H^k(\Omega)$, with $k = 0,1,2,\ldots$, is the set of functions on Ω, whose derivatives up to the order k are square integrable; in particular $H^0(\Omega) = L^2(\Omega)$. $H_0^k(\Omega)$ is the subset of $H^k(\Omega)$ of functions whose derivatives up to the order $k-1$ are zero at Γ (provided Γ smooth enough). Functions in $H_0^k(\Omega)$ may be considered as functions on the whole plane if we continue them by zero outside Ω; these continuations are in $H^k(\mathbb{R}^2)$. In $L^2(\Omega)$ the forms $(\cdot,\cdot)$ and $\|\cdot\|$ denote the usual inner product and norm

$$(u,v) := \iint_\Omega u(x,y)\bar{v}(x,y)\,dxdy, \qquad \|u\| := (u,u)^{\frac{1}{2}} ,$$

and in $H^1(\Omega)$ the vectorized forms $(\nabla u,\nabla v)$ and $\|\nabla u\|$ are defined by

$$(\nabla u,\nabla v) := (\partial_x u,\partial_x v) + (\partial_y u,\partial_y v), \qquad \|\nabla u\| := (\nabla u,\nabla u)^{\frac{1}{2}} .$$

The Laplace operator Δ is a formal differential operator, which may act on all functions in $H^2(\Omega)$; it is made to an (invertible) differential operator by restricting it to a suitable domain, e.g. $\Delta_{|D}$ is the restriction to the domain $D \subset H^2(\Omega)$. In general we shall denote the domain of an differential operator T by $\mathcal{D}(T)$ and its range by $\mathcal{R}(T)$.

The symbols ∂_x and ∂_y denote partial derivatives in the x and y-direction and ∂_n denotes the normal derivative in the direction of the outword drawn normal at the boundary.

2. THE EIGENVALUES AND RAYLEIGH'S QUOTIENT

Let T be a selfadjoint operator on a Hilbert space $\mathcal{H}$, let T be semibounded from below (i.e. $(Tu,u) \geq \gamma(u,u)$, $\gamma \in \mathbb{R}$) and let it have a compact inverse. As is well-known, cf. [12, ch. 3, § 6.8], the spectrum of T, $\sigma(T)$, consists of real isolated eigenvalues of finite multiplicity and the set of eigenfunctions corresponding to these eigenvalues is a complete orthonormal set in $\mathcal{H}$. Since T is semibounded with lower bound γ, no eigenvalue can be smaller than γ; hence we can arrange the eigenvalues in a non-decreasing sequence such that

$$\sigma(T) = \{\lambda_k \mid k \in \mathbb{N}\} \qquad \text{with} \qquad \lambda_{k+1} \geq \lambda_k, \ \forall_k \tag{2.1}$$

and such that each eigenvalue appears in the sequence as many times as its multiplicity is (the eigenvalue is counted according its multiplicity). To each eigenvalue λ_k corresponds an eigenfunction e_k such that $\{e_k \mid k \in \mathbb{N}\}$ is a complete orthonormal set in $\mathcal{H}$.

Since T is selfadjoint the inner product (Tu,u) is real for all $u \in \mathcal{D}(T)$. Expanding u in the eigenfunctions we find (if $u \neq 0$)

$$\frac{(Tu,u)}{(u,u)} = \sum_{k=1}^{\infty} \frac{\lambda_k (u,e_k)^2}{(u,u)} . \tag{2.2}$$

Clearly this quotient is minimal if $u = e_1$; it then yields the first eigenvalue. More general, if V is the span of k eigenfunctions, the maximum of the quotient (2.2) is just the largest eigenvalue connected to the eigenfunctions in V; clearly this maximum is minimized and equal to λ_k, if V is the span of the first k eigenfunctions. So it is plausible that λ_k satisfies the minimax characterization

$$\lambda_k = \min_{V \subset \mathcal{D}(T), \dim V = k} \ \max_{u \in V, u \neq 0} \frac{(Tu,u)}{(u,u)} . \tag{2.3}$$

The quotient (2.2) is called Rayleigh's quotient; the minimax characterization (2.3) is easily proved in the way suggested above, cf. [5, ch. 11].

Let L and M be the Laplace operator $-\Delta$ acting on smooth functions on a bounded domain Ω satisfying Dirichlet and Neumann boundary conditions respectively; these are well-known to be selfadjoint, semibounded and have a compact inverse. If $u \in \mathcal{D}(L)$ or $u \in \mathcal{D}(M)$ we can integrate the inner product $(-\Delta u,u)$ by parts and find

$$(2.4) \qquad (-\Delta u,u) = (u_x,u_x) + (u_y,u_y) \qquad (= \|\nabla u\|^2) .$$

If $\{\lambda_k \mid k \in \mathbb{N}\}$ and $\{\mu_k \mid k \in \mathbb{N}\}$ are the spectra of L and M, in which the eigenvalues are arranged in non-decreasing order, we find

$$\lambda_k = \min_{\substack{V\subset\mathcal{D}(L)\\ \dim V=k}} \max_{\substack{u\in V\\ u\neq 0}} \frac{\|\nabla u\|^2}{\|u\|^2} , \qquad \mu_k = \min_{\substack{V\subset\mathcal{D}(M)\\ \dim V=k}} \max_{\substack{u\in V\\ u\neq 0}} \frac{\|\nabla u\|^2}{\|u\|^2} .$$

We see from this formula that minima do not change if V ranges over the closures of $\mathcal{D}(L)$ and $\mathcal{D}(M)$ with respect to the norm $\|u\| + \|\nabla u\|$. These closures are $H_0^1(\Omega)$ and $H^1(\Omega)$ respectively. We conclude that the trial space V in the characterization of μ_k ranges over a larger set than it does in the characterization of λ_k, hence the minimum over the larger set may be smaller. So we find

$$(2.5) \qquad \mu_k < \lambda_k, \qquad \forall_{k\in\mathbb{N}} .$$

Thus we see how the Rayleigh quotient characterization (2.2) of the eigenvalues may be a suitable tool for comparing eigenvalues of differential operators.

3. REDUCTION TO ZERO-th ORDER

We shall study the spectral properties of the second order elliptic operator

$$(3.1) \qquad L_\varepsilon := -\varepsilon\Delta + p \quad \text{on} \quad D := \mathcal{D}(L_\varepsilon) := H_0^1(\Omega) \cap H^2(\Omega) ,$$

where Ω is a bounded domain satisfying the "cone condition", where ε is a small parameter and where p is a bounded continuous function on Ω.

It is well-known, cf. [19], that the operator Δ on D is an unbounded operator with a compact inverse and, hence, that its spectrum is discrete (consists of isolated eigenvalues only). Since the operator "multiplication by p" is a bounded operator, the sum $-\varepsilon\Delta + p$ again has a compact inverse and an unbounded discrete spectrum for each $\varepsilon > 0$, cf. [12, ch. 4, th. 1.6]. However, the formal limit operator (for $\varepsilon \to +0$) "multiplication by p" has a bounded purely continuous spectrum (provided p not constant), which is equal to the range of the function $p: \Omega \to \mathbb{R}$. It looks natural that this range $\mathcal{R}(p)$ is the limit of $\sigma(L_\varepsilon)$ in some sense. Indeed, each individual eigenvalue eventually becomes absorbed in $\mathcal{R}(p)$ for $\varepsilon \to +0$; however, $\mathcal{R}(p)$ is not the limiting set of $\sigma(L_\varepsilon)$ in the sense that it contains all points of accumulation of the union $\bigcup_{\varepsilon>0} \sigma(L_\varepsilon)$, since $\sigma(L_\varepsilon)$ extends to $+\infty$ for every $\varepsilon > 0$ and becomes more and more dense everywhere on the real axis above the minimum of p, as ε tends to zero. The growth beyond bound of the density suggest that it is impossible to describe the limiting behaviour of each individual eigenvalue; apparently it is better to compute a more global quantity, namely the asymptotic density for $\varepsilon \to +0$.

We shall first review some properties of the operator $-\Delta$ on the domain D (cf. 3.1). Let the spectrum of $-\Delta_{|D}$ be the set

$$\sigma(-\Delta_{|D}) = \{\mu_k \mid k \in \mathbb{N}\} \qquad (\mu_{k+1} \geq \mu_k) \tag{3.2}$$

in which the eigenvalues are ordered in non-decreasing sense and are counted according to their multiplicity. They satisfy, cf. [5, ch. 11],

$$\mu_1 = \min_{u \in H_0^1(\Omega), \|u\|=1} \|\nabla u\|^2 > 0 \ , \tag{3.3}$$

$$\#\{\mu \in \sigma(-\Delta_{|D}) \mid \mu \leq \lambda\} = \frac{A\lambda}{4\pi} + \mathcal{O}(\lambda^{\frac{1}{2}}\log \lambda)^{*)} \qquad (\lambda \to \infty) \ , \tag{3.4}$$

where A stands for the area of Ω and where $\#\{\cdot\}$ denotes the number of elements of the set $\{\cdot\}$. Formula (3.4) is proved by sandwiching the eigenvalues between the eigenvalues of two operators whose spectra (and the densities thereof) are known. We shall use the same idea in the computation of the asymptotic density of $\sigma(L_\varepsilon)$. From (3.2-4) we infer that, if $p \equiv p_0$ is a constant, the spectrum is

$$\sigma(-\varepsilon\Delta_{|D} + p_0) = \{\varepsilon\mu_k + p_0 \mid k \in \mathbb{N}\} \tag{3.5}$$

and that the number of its elements below λ satisfies the estimate

$$\#\{\mu \in \sigma(-\varepsilon\Delta_{|D} + p_0) \mid \mu \leq \lambda\} = \frac{A}{4\pi\varepsilon}(\lambda - p_0)(1 + \mathcal{O}(\varepsilon^{\frac{1}{2}}\log \varepsilon)) \tag{3.6}$$

for $\varepsilon \to +0$, provided $\lambda > p_0$ (λ fixed).

Let us denote the eigenvalues of L_ε for $\varepsilon > 0$ by $\lambda_k(\varepsilon)$ with $k \in \mathbb{N}$, such that $\lambda_{k+1} \geq \lambda_k$ and such that each eigenvalue is counted according to its multiplicity; hence

*) In fact the remainder is of the order $\mathcal{O}(\lambda^{\frac{1}{2}})$.

(3.7) $$\sigma(L_\varepsilon) = \{\lambda_k(\varepsilon) \mid k \in \mathbb{N}\} ,$$

and the eigenvalues satisfy the minimax characterization (2.3),

(3.8) $$\lambda_k(\varepsilon) = \min_{V\subset H_0^1(\Omega),\dim V=k} \; \max_{u\in V,\|u\|=1} (\varepsilon\|\nabla u\|^2 + (pu,u)) .$$

Denoting by p_{min} and p_{max} the minimum and maximum of p,

(3.9) $$p_{min} = \min_{(x,y)\in\bar\Omega} p(x,y), \qquad p_{max} = \max_{(x,y)\in\bar\Omega} p(x,y) ,$$

we easily see from (3.8) that the eigenvalues of L_ε are sandwiched between those of $-\varepsilon\Delta_{|D} + p_{min}$ and $-\varepsilon\Delta_{|D} + p_{max}$. Hence they satisfy

(3.10) $$\varepsilon\mu_k + p_{min} \le \lambda_k(\varepsilon) \le \varepsilon\mu_k + p_{max}$$

and we conclude from this that each eigenvalue eventually becomes absorbed in the set $R(p)$ as ε tends to zero. Moreover, $\lambda_k(\varepsilon)$ decreases monotonically if ε decreases, as can be seen from (3.8) in the following way. Let V_δ be the total eigenspace belonging to the first k eigenvalues of L_δ. If $\varepsilon < \delta$ we find from (3.3-8)

(3.11) $$\begin{aligned}\lambda_k(\varepsilon) &\le \max_{u\in V_\delta,\|u\|=1} (\varepsilon\|\nabla u\|^2 + (pu,u)) \le \\ &\le \max_{u\in V_\delta,\|u\|=1} (\delta\|\nabla u\|^2 + (pu,u)) - (\delta-\varepsilon) \min_{u\in V_\delta,\|u\|=1} \|\nabla u\|^2 \le \\ &\le \lambda_k(\delta) - (\delta-\varepsilon)\mu_1 < \lambda_k(\delta) .\end{aligned}$$

Summing up we have shown concerning the individual eigenvalues:

THEOREM 1. *Each eigenvalue $\lambda_k(\varepsilon)$ of L_ε is a (strictly) increasing function of ε which satisfies*

(3.12) $$\lambda_k(\varepsilon) \ge p_{min}, \qquad \lim_{\varepsilon\to 0} \lambda_k(\varepsilon) \le p_{max} .$$

For an asymptotic estimate of the global quantity, the density, or better, the number of eigenvalues below λ,

(3.13) $$n_\varepsilon(\lambda) := \#\{\lambda_k(\varepsilon) \in \sigma(L_\varepsilon) \mid \lambda_k(\varepsilon) \le \lambda\} ,$$

the estimates (3.10) are far too rough (except if p is constant, cf. (3.6)). Therefore we construct domains for the formal operator $-\varepsilon\Delta + p$, which are smaller and larger than D, for which we can compute the numbers of eigenvalues below λ, and by which we can estimate the eigenvalues of L_ε. Garabedian [5, ch. 11] employes the same idea to prove formula (3.4).

Let us choose a mesh-width h and let us cover Ω by the rectangular grid G,

$$(3.14) \qquad G := \{(x,y) \in \mathbb{R}^2 \mid x = ih \text{ or } y = jh,\ i,j \in \mathbb{Z}\} ,$$

which cuts the plane in the squares S_{ij},

$$S_{ij} := \{(x,y) \in \mathbb{R}^2 \mid ih < x < ih + h,\ jh < y < jh + h\} ,$$

and let p_{ij} and P_{ij} be the infimum and the supremum of p over this square,

$$p_{ij} := \inf\{p(x,y) \mid (x,y) \in R_{ij}\},\ P_{ij} := \sup\{p(x,y) \mid (x,y) \in R_{ij}\}.$$

Let I be the set of indices of squares contained in Ω, let Ω_I be their union, let J be the set of squares which have an non-empty intersection with Ω and let Ω_J be their union; in formulae

$$I := \{(i,j) \mid R_{ij} \subset \Omega\}, \qquad \Omega_I := \bigcup_{(i,j)\in I} R_{ij} ,$$

$$J := \{(i,j) \mid R_{ij} \cap \Omega \neq \emptyset\}; \qquad \Omega_J := \bigcup_{(i,j)\in J} R_{ij} .$$

On the sets Ω_I and Ω_J we define the function spaces D_I and D_J,

$$D_I := \{u \in H_0^1(\Omega) \mid u = 0 \text{ on the grid G and outside } \Omega_I\} ,$$

$$D_J := \{u \in L^2(R) \mid u \in H^1(S_{ij}) \text{ for all } i,j \in J\} .$$

Clearly the trial space $H_0^1(\Omega)$ (whose elements may be continued outside Ω by zero) is contained in D_J and it contains D_I. Hence, if we replace this trial space in the minimax characterization (3.8) by D_I and D_J, we obtain an upper and a lower estimate respectively for $\lambda_k(\varepsilon)$.

Let us now consider the sets N_ε and $\tilde{N}_\varepsilon$ of numbers defined by minimax characterizations:

$$N_\varepsilon := \{\nu_k(\varepsilon) \mid \nu_k = \min_{V\subset D_I,\dim V=k}\ \max_{u\in V,\|u\|=1} (\varepsilon\|\nabla u\|^2 + (pu,u)),\ k \in \mathbb{N}\},$$

$$\tilde{N}_\varepsilon := \{\tilde{\nu}_k(\varepsilon) \mid \tilde{\nu}_k = \min_{V\subset D_J,\dim V=k}\ \max_{u\in V,\|u\|=1} (\varepsilon\|\nabla u\|^2 + (pu,u)),\ k \in \mathbb{N}\}.$$

Since D_I is the linear hull of the set of spaces $H_0^1(S_{ij})$ where (i,j) ranges over I (provided we continue the elements of $H_0^1(S_{ij})$ by zero outside S_{ij}), N_ε can be recognized as the union of the spectra of the restrictions of L_ε to $H_0^1(R_{ij}) \cap H^2(R_{ij})$. The eigenvalues of a restriction can be estimated from above by the eigenvalues of the operator $-\varepsilon\Delta + P_{ij}$ on $H_0^1(R_{ij}) \cap H^2(R_{ij})$ and the latter can be computer exactly. From formula (3.6) we find that the number of eigenvalues below λ of the latter operator is equal to

$$(3.15) \qquad \frac{h^2}{4\pi\varepsilon}(\lambda - P_{ij})(1 + \mathcal{O}(\varepsilon^{\frac{1}{2}}\log \varepsilon)) \ ,$$

provided $\lambda > P_{ij}$, and is zero otherwise. Summing this quantity up over all indices $(i,j) \in I$ we find an upper Riemann sum corresponding to the partition induced by G for the integral

$$(3.16) \qquad \frac{1}{4\pi\varepsilon} \iint_{\Omega} (\lambda - p(x,y)) \wedge 0 \ dxdy \ (1 + \mathcal{O}(\varepsilon^{\frac{1}{2}}\log \varepsilon)) \ ,$$

where $f \wedge 0$ denotes the function which is equal to f in all points where f is positive and which is zero otherwise.

Likewise $\tilde{N}_\varepsilon$ can be recognized as the joint spectrum of the restrictions of $-\varepsilon\Delta + p$ to the sets $\{u \in H^2(R_{ij}) \mid \partial_n u|_\Gamma = 0\}$, i.e. to functions on R_{ij} which satisfy a Neumann type of boundary condition. The eigenvalues of these restricted operators can be estimated from below by the eigenvalues the operators $-\varepsilon\Delta + p_{ij}$ on $\{u \in H^2(\Omega) \mid \partial_n u|_\Gamma = 0\}$. The number of the eigenvalues satisfies the estimate (3.15) with P_{ij} replaced by p_{ij}, which produces a lower Riemann sum for the same integral (3.16). By refining of the grid we find the limiting density below λ:

THEOREM 2. *The number of eigenvalues* $n_\varepsilon(\lambda)$ *of* L_ε *below* λ *satisfies the asymptotic formula for* $\varepsilon \to +0$

$$(3.17) \qquad n_\varepsilon(\lambda) = \frac{1}{4\pi\varepsilon} \iint_{\Omega} (\lambda - p(x,y)) \wedge 0 \ dxdy \ (1 + \mathcal{O}(\varepsilon^{\frac{1}{2}}\log \varepsilon)) \ .$$

The theorems 1 & 2 admit considerable generalizations. We can derive analogous estimates for a bounded domain (satisfying the cone condition) in any finite dimensional space. If $\Omega \subset \mathbb{R}^n$, the number of eigenvalues below λ is of the order $O(\varepsilon^{-n/2})$. Furthermore, we may consider perturbations by any symmetric second order elliptic operator $L_\varepsilon = \varepsilon \sum_{ij} \partial_i a_{ij} \partial_j + p$ with $a_{ij} = a_{ji}$. For this more general operator the grid (3.14) is not suitable and we have to choose a more sophisticated grid which follows the coordinate hyperplanes of a transformation which diagonalizes the symmetric bilinear form $\sum a_{ij}\xi_i\eta_j$ with $\xi,\eta \in \mathbb{R}^n$ for every $x \in \Omega$. Alternatively we may employ an analogue of Gårding's method, cf. [6]. In this more general case the integrand in formula (3.17) is multiplied by a constant times the volume of the ellipsoid $\sum a_{ij}\xi_i\xi_j < 1$, which is equal to a multiple of $(\det(a_{ij}))^{-\frac{1}{2}}$, cf. [6, formula (0.6)].

4. REDUCTION TO FIRST ORDER

In this section we shall study the spectral behaviour of the second order elliptic operator L_ε which reduces to a first order operator for $\varepsilon \to +0$,

$$(4.1) \qquad L_\varepsilon := -\varepsilon\Delta + p\partial_x + q\partial_y \quad \text{on} \quad \mathcal{D}(L_\varepsilon) := D := H_0^1(\Omega) \cap H^2(\Omega) ,$$

where Ω again is an open domain in $\mathbb{R}^2$ satisfying the cone property, and where p and q are smooth functions on $\bar{\Omega}$. The striking difference between the type of operators (4.1) and the operators we study in the sections 3 and 5 is, that an operator of type (4.1) is essentially non-symmetric (if p and q are real) because of the first order part $p\partial_x + q\partial_y$. This implies that the spectrum need not be real and that the eigenvalues do not satisfy the minimax property. The only thing we know in advance is that the spectrum of L_ε consists of isolated eigenvalues of finite multiplicity for each $\varepsilon > 0$ (because $\Delta_{|D}$ has a compact inverse and $p\partial_x + q\partial_y$ is relatively compact with respect to Δ, cf. [12, ch. 4, th. 1.6]. The limiting behaviour for $\varepsilon \to +0$ may vary widely depending on the characteristics of the formal limit operator $p\partial_x + q\partial_y$. The characteristics, cf. [5, ch. 2], are the integral curves of the system of autonomous equations

$$(4.2) \qquad \frac{dx}{dt} = p(x,y), \; \frac{dy}{dt} = q(x,y) ,$$

and along them $p\partial_x + q\partial_y$ reduces to d/dt.

4.1. *The spectrum recedes to infinity* for $\varepsilon \to +0$, if $\bar{\Omega}$ does not contain a critical point of (4.2) and if a (smooth) transformation $(x,y) \to (s,t)$ exists such that $p\partial_x + q\partial_y$ is transformed in ∂_t. For the operator $-\varepsilon\Delta + \partial_y$ on D this follows from the estimate, valid for all $u \in D$ and $\alpha \in \mathbb{R}$:

$$(4.3) \qquad \|-\varepsilon\Delta u + \partial_y u - \lambda u\| \, \|e^{-\alpha y}u\| \geq \mathit{Re}(-\varepsilon\Delta u + \partial_y u - \lambda u, \; e^{-\alpha y}u) =$$

$$= \varepsilon(\nabla u, e^{-\alpha y}\nabla u) + (\tfrac{1}{2}\alpha - \tfrac{1}{2}\varepsilon\alpha^2 - \mathit{Re}\,\lambda)(u, e^{-\alpha y}u) .$$

The right-hand side is strictly positive for all non-zero $u \in D$ if $\frac{1}{2}\alpha - \frac{1}{2}\varepsilon\alpha^2 - \mathit{Re}\,\lambda > 0$. Hence no eigenvalue can satisfy $\mathit{Re}\,\lambda = \frac{1}{2}\alpha - \frac{1}{2}\varepsilon\alpha^2 \leq \frac{1}{8\varepsilon}$ for any $\alpha \in \mathbb{R}$. Extending the same argument to the more general case we find:

THEOREM 3. *If* $p^2 + q^2 \geq \gamma > 0$ *for all* $(x,y) \in \Omega$ *and if* Ω *is simply connected (i.e. without holes), (or better: if* $\exists_{\varphi \in C^2(\Omega)}$ s.t. $|p\varphi_x + q\varphi_y| \geq \gamma > 0$ *uniformly in* Ω*), then a positive constant* C *not depending on* ε *exists, such that*

$$\lambda \in \sigma(L_\varepsilon) \Rightarrow \operatorname{Re} \lambda \geq C/\varepsilon \ . \tag{4.4}$$

NOTE: For an analysis of the steady state problem Lu = f in this case see [4].

4.2. *A discrete limit set* occurs in the case where Ω contains critical points of the system (4.2) and if the functional matrix $(\nabla p, \nabla q)$ has non-zero non-imaginary eigenvalues at those critical points. We do not intend to prove this fact here or even to give an indication how such could be done in general, since the proofs we know are fairly complicated, cf. [8] and [13]. We shall give a proof only in the very simple case, where L_ε is the operator (with $\alpha \neq 0$ and $\beta \neq 0$)

$$L_\varepsilon := -\varepsilon\Delta + \alpha x \partial_x + \beta y \partial_y, \qquad \mathcal{D}(L_\varepsilon) = D \ . \tag{4.5}$$

Its analysis depends heavily on the analysis for the analogous one-dimensional operator T_ε on the real interval $(-a,a)$, with $a > 0$:

$$T_\varepsilon := -\varepsilon \frac{d^2}{dx^2} + x \frac{d}{dx} \quad \text{on} \quad \mathcal{D}(T_\varepsilon) := H_0^1(-a,a) \cap H^2(-a,a) \ . \tag{4.6}$$

Therefore we shall examine first the spectrum of T_ε.

As is well-known, the ordinary differential operator T_ε can be made selfadjoint by the Liouville transformation

$$u \to v \exp(-\tfrac{1}{2} \int x \, dx/\varepsilon) \ .$$

Equivalently we may choose in $L^2(-1,1)$ the new inner product

$$[u,v]_\varepsilon := \int_{-a}^{a} u(x)\bar{v}(x) \exp(-\tfrac{1}{2}x^2/\varepsilon)\, dx = (u, e^{-x^2/2\varepsilon} v) \tag{4.7}$$

with respect to which T_ε is selfadjoint,

$$[T_\varepsilon u, v]_\varepsilon = [u, T_\varepsilon v]_\varepsilon = [u', v']_\varepsilon, \qquad \forall_{\varepsilon > 0}, \ \forall_{u,v \in D} \ . \tag{4.8}$$

For each $\varepsilon > 0$ the norm connected with $[\cdot,\cdot]_\varepsilon$ is equivalent to the original norm (although <u>not</u> uniformly!),

$$[\![u]\!]_\varepsilon \leq \|u\| \leq \exp(a^2/2\varepsilon) [\![u]\!]_\varepsilon, \qquad [\![u]\!]_\varepsilon := [u,u]_\varepsilon^{\frac{1}{2}} \ , \tag{4.9}$$

hence the induced topologies and the spectra of T_ε are the same for both inner products in $L^2(-a,a)$.

As a consequence of (4.8) the eigenvalues of T_ε satisfy again the minimax characterization

$$\lambda_k(\varepsilon) = \min_{V\subset H_0^1(-a,a),\dim V=k+1} \; \max_{u\in V,\|u\|=1} [\![u']\!]_\varepsilon^2 \tag{4.10}$$

if $\sigma(T_\varepsilon) = \{\lambda_k(\varepsilon) \mid k = 0,1,2,\ldots\}$ and if $\lambda_{k+1} > \lambda_k$. The eigenvalues and eigenfunctions are solutions of the equation

$$T_\varepsilon u - \lambda u = -\varepsilon u'' + xu' - \lambda u = 0 \ , \tag{4.11}$$

which reduces to Hermite's equation by the transformation $\xi = x/\sqrt{2\varepsilon}$. The only solutions of (4.11) which are not exponentially increasing are the Hermite polynomials H_k,

$$(T_\varepsilon - k)H_k(x/\sqrt{2\varepsilon}) = 0, \quad \forall_{k=0,1,2,\ldots} \ . \tag{4.12}$$

Those functions form a suitable trial set in (4.10), if we plunge them in $H_0^1(-a,a)$ by multiplying them by the cut-off function $\psi \in C^\infty(\mathbb{R})$ which satisfies

$$\psi(x) = 1 \quad \text{if} \quad |x| \le \tfrac{1}{2}a \quad \text{and} \quad \psi(x) = 0 \quad \text{if} \quad |x| \ge a \ .$$

Simple computation shows that a constant $C_k > 0$ exists such that

$$[(T_\varepsilon - k)(H_k(x/\sqrt{\varepsilon})\psi(x)),\ H_k(x/\sqrt{\varepsilon})\psi(x)]_\varepsilon \le C_k\varepsilon^{-k}e^{-a^2/2\varepsilon} \ . \tag{4.13}$$

Hence, choosing V as the linear span of the first $k+1$ of those functions, we find the upper bound

$$\lambda_k(\varepsilon) \le k + \mathcal{O}(\varepsilon^{-k}e^{-a^2/8\varepsilon}) \ . \tag{4.14}$$

A lower bound for $\lambda_k(\varepsilon)$ we find by enlarging in (4.10) the trial space $H_0^1(-a,a)$ to $H^1(\mathbb{R})$. In this larger space the Hermite polynomials $H_k(x/\sqrt{2\varepsilon})$ form a complete orthogonal set with respect to the inner product (4.7); since these polynomials are exactly the solution of (4.12), the eigenvalues of $-\varepsilon d^2/dx^2 + xd/dx$ on $H^2(\mathbb{R})$ are exactly the non-negative integers. Hence $\lambda_k(\varepsilon)$ is bounded from below by k,

$$k < \lambda_k(\varepsilon) \le k + \mathcal{O}(\varepsilon^{-k}e^{-a^2/8\varepsilon}) \ . \tag{4.15}$$

REMARKS. 1. The eigenvalues of the adjoint operator T_ε^*,

$$(4.16) \qquad T_\varepsilon^* = -\varepsilon \frac{d^2}{dx^2} - x \frac{d}{dx} - 1$$

are equal to those of T_ε and hence satisfy the estimate (4.15).

2. This analysis in one dimension can easily be generalized to arbitrary operators in which the coefficient of the first order part has simple zeros, cf. de Groen [10]. The problem is known as the singularly perturbed turning point problem, cf. Ackerberg & O'Malley [1].

By analogy to (4.7-8) the two-dimensional operator L_ε becomes selfadjoint with respect to the inner product

$$(4.17) \qquad [u,v]_\varepsilon := (u, e^{-(x^2+\alpha y^2)/2\varepsilon} v) \ .$$

If Ω is the square $(-a,a) \times (-a,a)$ and if $\alpha > 0$ and $\beta > 0$ we find by separation of variables $L_\varepsilon = \alpha T_{\varepsilon/\alpha} \otimes \beta T_{\varepsilon/\beta}$ such that

$$(4.18a) \qquad \sigma(L_\varepsilon) = \{\alpha\lambda_i(\varepsilon/\alpha) + \beta\lambda_j(\varepsilon/\beta) \mid i,j = 0,1,2,\ldots\} \ ;$$

if $\beta < 0 < \alpha$ we find $L_\varepsilon = \alpha T_{\varepsilon/\alpha} \otimes \beta(T^*_{\varepsilon/\beta} + \beta)$, such that (4.16) implies

$$(4.18b) \qquad \sigma(L_\varepsilon) = \{\alpha\lambda_i(\varepsilon/\alpha) + \beta\lambda_j(\varepsilon/\beta) + \beta \mid i,j = 0,1,\ldots\} \ ,$$

and if both are negative we find an analogous set. If Ω is a more general domain, squares with edges a and b $(0 < a < b)$ exist such that

$$\Omega_i = (-a,a) \times (-a,a) \subset \Omega \subset (-b,b) \times (-b,b) = \Omega_e \ .$$

Hence, the minimax characterization of the eigenvalues with respect to the inner product (4.17) implies that the eigenvalues of L_ε are sandwiched by those on the inscribed and circumscribed squares Ω_i and Ω_e. Thus we have derived:

THEOREM 4. *The eigenvalues of* L_ε *are real and we can arrange them in non-decreasing order (counting multiplicity) such that*

$$\sigma(L_\varepsilon) = \{\mu_j(\varepsilon) \mid j \in \mathbb{N}\}, \qquad \mu_{j+1} \geq \mu_j \quad \forall_j \ .$$

If $\{\nu_j \mid j \in \mathbb{N}\}$ *is a non-decreasing reordering of the set* $\{\alpha k + \beta \ell \mid k,\ell = 0,1,2,\ldots\}$, *then the eigenvalues have the limits*

$$(4.19)\qquad \lim_{\varepsilon\to+0} \mu_j(\varepsilon) = \begin{cases} \nu_j, & \text{if } \alpha > 0,\ \beta > 0, \\ \nu_j + \alpha, & \text{if } \alpha < 0 < \beta, \\ \nu_j + \beta, & \text{if } \alpha > 0 > \beta, \\ \nu_j + \alpha + \beta, & \text{if } \alpha < 0,\ \beta < 0. \end{cases}$$

REMARK. Formula (4.19) remains valid for more general operators of type (4.1) if the Jacobian matrix of (p,q) at the critical point has the non-zero real eigenvalues α and β, cf. [8] and [13]. Moreover, the proofs given there are easily generalized to spaces Ω of higher dimension.

4.3. *A dense limit set* may occur in the case where the eigenvalues of the Jocobian matrix of (p,q) at a critical point are purely imaginary or zero. We shall give an example of the first type; for an example of the second type see [11].

Let Ω be a disk (or annulus) around the origin and let L_ε be the operator

$$(4.20)\qquad L_\varepsilon := -\varepsilon\Delta + x\partial_y - y\partial_x = -\frac{\varepsilon}{r}\,\partial_r r\partial_r - \varepsilon r^{-2}\partial_\varphi^2 + \partial_\varphi$$

on $\mathcal{D}(L_\varepsilon) := D := H_0^1(\Omega) \cap H^2(\Omega)$, where (r,φ) are the usual polar coordinates. This operator L_ε happens to be normal, i.e. $L_\varepsilon^* L_\varepsilon = L_\varepsilon L_\varepsilon^*$, hence its "real" and "imaginary" parts

$$\tfrac{1}{2}L_\varepsilon + \tfrac{1}{2}L_\varepsilon^* = \varepsilon\Delta \qquad\text{and}\qquad \tfrac{1}{2}i(L_\varepsilon^* - L_\varepsilon) = -i\partial_\varphi\ ,$$

commute and are selfadjoint. The set of integers $\mathbb{Z}$ is the spectrum of the "imaginary" part and the eigenspace corresponding to the eigenvalue k is $\{v(r)e^{ik\varphi}\}$, where $v \in D$ depends on r only. This eigenspace is invariant under the "real" part of L_ε, hence on this eigenspace the eigenvalue equation $L_\varepsilon u = \lambda u$ reduces to

$$-\frac{\varepsilon}{r}\,\partial_r r\partial_r v + \varepsilon r^{-2}k^2 v - ikv = \lambda v\ ,$$

which is Bessel's equation. By analogy to the problem of section 3 it is easily seen that the part of the spectrum of L_ε due to the eigenspace $\{v(r)e^{ik\varphi}\}$ of $-i\partial_\varphi$ becomes dense in the halfline $\{\mu - ik \mid \mu \in \mathbb{R}^+\}$ for $\varepsilon \to +0$. We conclude that $\sigma(L_\varepsilon)$ becomes dense in the union of all these halflines.

5. REDUCTION TO SECOND ORDER

In this section we shall study the behaviour of the eigenvalues of the fourth order singularly perturbed differential operator L_ε acting on functions on a bounded domain Ω with a smooth boundary Γ,

$$(5.1)\qquad L_\varepsilon := \varepsilon\Delta^2 - \Delta \quad \text{on} \quad \mathcal{D}(L_\varepsilon) := \{u \in H^4(\Omega) \mid u_{|\Gamma} = \partial_n u_{|\Gamma} = 0\} .$$

The formal limit operator for $\varepsilon \to +0$ is $-\Delta$; it is not natural to attach to it more than <u>one</u> boundary condition, hence we define L_0 by

$$(5.2)\qquad L_0 := -\Delta \quad \text{on} \quad \mathcal{D}(L_0) := \{u \in H^2(\Omega) \mid u_{|\Gamma} = 0\} .$$

The spectra of L_ε and L_0 are discrete sets and consist of (positive) eigenvalues of finite multiplicity only. Arranging the eigenvalues in non-decreasing order and counting them according to their multiplicity we find

$$(5.3)\qquad \sigma(L_\varepsilon) = \{\lambda_k(\varepsilon) \mid k \in \mathbb{N}\}, \qquad \text{with } \lambda_{k+1}(\varepsilon) \geq \lambda_k(\varepsilon) ,$$

$$(5.4)\qquad \sigma(L_0) = \{\lambda_{k,0} \mid k \in \mathbb{N}\}, \qquad \text{with } \lambda_{k+1,0} \geq \lambda_{k,0} .$$

Assuming that the eigenvalues and eigenfunctions of L_0 are known, we shall construct asymptotic series expansions for the eigenvalues and the corresponding eigenfunctions of L_ε; in the case where Ω is the unit circle, we find explicitly for the eigenvalue $\mu_{k\ell}$

$$(5.5)\qquad \mu_{k\ell}(\varepsilon) = \alpha_{k\ell}^2 + \tfrac{1}{2}\varepsilon^{\frac{1}{2}}\alpha_{k\ell}^2 + \mathcal{O}(\varepsilon), \qquad \varepsilon \to +0,\ \ell \in \mathbb{N},\ k \in \mathbb{Z} ,$$

where $\alpha_{k\ell}$ is the ℓ-th zero of the k-th Bessel function J_k. The construction of the eigenfunctions is performed by the method of "matched asymptotic expansions" in combination with an analogue of the method of "suppression of secular terms" (in celestial mechanics) by which in each step a term of the asymptotic series of $\lambda_k(\varepsilon)$ is determined.

The methods we use and the results we obtain for the operator (5.1) can be generalized easily to general selfadjoint elliptic operators of the type $L_\varepsilon = \varepsilon L_{2m+2k} + L_{2m}$ on a bounded set in n dimensions, where L_{2j} is a symmetric uniformly elliptic formal differential operator of order 2j, which is bounded from below on the domain of definition of L_ε. For the eigenfunctions and eigenvalues we then find asymptotic power series in $\varepsilon^{1/k}$, which start with the eigenfunctions and eigenvalues of L_{2j} whose domain is restricted by the j boundary conditions of L_ε, which are of lowest order (provided their order is smaller than 2j).

Greenlee studies in [7] the same problems. He derives for (5.1) the weaker result

$$(5.6) \qquad \lambda_k(\varepsilon) = \lambda_{k,0} + \mathcal{O}(\varepsilon^{\tau}), \qquad \varepsilon \to +0,\ \tau \in [0,\tfrac{1}{2}) ,$$

by interpolation of Hilbert spaces. His method does not provide a method for computing the second (let be higher order) terms of the asymptotic power series of the eigenvalues and the corresponding eigenfunctions. Moreover, Greenlee's method uses much deeper functional analytic tools than the method we shall employ. Moser [15] studies the analogue of (5.1) in one dimension; the method displayed here can be applied to those problems too.

REMARK. The inversion of the sign of ε in (5.1) causes a dramatic change in the spectrum; it becomes dense on the whole real axis for $\varepsilon \to -0$.

5.1. *A lower bound for the eigenvalues* is derived by a comparison analogous to (2.5). The eigenvalues of L_ε and L_0 satisfy the characterization, cf. (2.3),

$$(5.7) \qquad \lambda_k(\varepsilon) = \min_{V\subset H_0^2(\Omega),\dim V=k} \ \max_{u\in V,\|u\|=1} \ \varepsilon\|\Delta u\|^2 + \|\nabla u\|^2, \qquad \varepsilon > 0,$$

$$(5.8) \qquad \lambda_{k,0} = \min_{V\subset H_0^1(\Omega),\dim V=k} \ \max_{u\in V,\|u\|=1} \ \|\nabla u\|^2 .$$

If we enlarge in (5.7) the domain H_0^2, over which the subspace V ranges, and if we define the numbers $\nu_k(\varepsilon)$ by

$$(5.9) \qquad \nu_k(\varepsilon) := \min_{V\subset H^2\cap H_0^1(\Omega),\dim V=k} \ \max_{u\in V,\|u\|=1} \ \varepsilon\|\Delta u\|^2 + \|\nabla u\|^2 ,$$

(i.e. we have skipped the boundary condition $\partial_n u_{|\Gamma} = 0$), we find that $\nu_k(\varepsilon)$ is the k-th eigenvalue of the operator N_ε,

$$(5.10) \qquad N_\varepsilon := \varepsilon\Delta^2 - \Delta \quad \text{on} \quad \mathcal{D}(N_\varepsilon) := \{u \in H^4(\Omega) \mid u_{|\Gamma} = \Delta u_{|\Gamma} = 0\} .$$

It is clear that this operator, which differs from L_ε only in its boundary conditions, satisfies $N_\varepsilon = \varepsilon L_0^2 + L_0$. Hence, by the spectral mapping theorem we find

$$(5.11) \qquad \nu_k(\varepsilon) = \lambda_{k,0} + \varepsilon\lambda_{k,0}^2 .$$

Since the minimum in (5.9) is taken over a larger set than the minimum in (5.7), the eigenvalue $\lambda_k(\varepsilon)$ is not smaller than $\nu_k(\varepsilon)$; hence

(5.12) $\quad \lambda_k(\varepsilon) \geq \lambda_{k,0} + \varepsilon\lambda_{k,0}^2$.

5.2. *The construction of formal expansions* of eigenvalues and eigenfunctions for L_ε is analogous to the construction given by Besjes [2] and Vishik & Lyusternik [17] for the Dirichlet problem $L_\varepsilon u = f$, if we add a formal power series expansion for the unknown eigenvalue.

We start from the assumption that an asymptotic approximation of an eigenfunction u consists of an outer expansion and of a boundary layer expansion in a neighbourhood of Γ. For the latter we define local coordinates (ρ,s) in a neighbourhood of the boundary Γ such that $\rho(x,y)$ represents the distance from (x,y) to the boundary Γ and s the arc length along Γ, cf. [4, formula 3.13]. If Γ is smooth a constant $\rho_0 > 0$ exists such that the mapping $(x,y) \to (\rho,s)$ is one to one from a strip along Γ to the strip $0 < \rho < \rho_0$ (modulo the arc length of Γ). In order to find the boundary layer terms, we stretch the distance ρ to the boundary by such a power of ε that the lowest order parts of $\varepsilon\Delta^2$ and Δ in the stretched variable are of equal order (with respect to ε). The choice $t := \varepsilon^{-\frac{1}{2}}\rho$ will do, for then we find the formal expansion

(5.13) $$\varepsilon\Delta^2 - \Delta = \frac{1}{\varepsilon}(\partial_t^4 - \partial_t^2) + \varepsilon^{-\frac{1}{2}}M_1 + M_2 + \varepsilon^{\frac{1}{2}}M_3 + \dots ,$$

where M_1, M_2, etc. are determined from the transformation $(x,y) \to (\varepsilon^{\frac{1}{2}}t,s)$ and the expansion of the coefficients in powers of $\varepsilon^{\frac{1}{2}}$. This expansion of L_ε suggest that the formal series for the eigenvalue λ and the eigenfunction u will be series in half-integral powers of ε too,

(5.14) $$u \sim \sum_{j=0}^{\infty} \varepsilon^{\frac{1}{2}j} v_j(x,y) + \varepsilon^{\frac{1}{2}} \sum_{j=0}^{\infty} \varepsilon^{\frac{1}{2}j} w_j(t,s) ,$$

(5.15) $$\lambda \sim \sum_{j=0}^{\infty} \varepsilon^{\frac{1}{2}j} \alpha_j .$$

Inserting these formal expansions in the eigenvalue equation

(5.16) $$L_\varepsilon u = \lambda u, \qquad u_{|\Gamma} = \frac{\partial u}{\partial n}_{|\Gamma} = 0 ,$$

and collecting equal powers of ε, we find the set of differential equations

(5.17a) $\quad -\Delta v_0 - \alpha_0 v_0 = 0$,

(5.17b) $\quad -\Delta v_1 - \alpha_0 v_1 = \alpha_1 v_0$, (5.17c) $\quad -\Delta v_j - \alpha_0 v_j = \sum_{m=1}^{j} \alpha_m v_{j-m} - \Delta^2 v_{j-2}, \qquad (j \geq 2)$,

(5.18a) $\quad (\partial_t^4 - \partial_t^2) w_0 = 0$,

(5.18b) $\quad (\partial_t^4 - \partial_t^2) w_1 = -M_1 w_0$,

(5.18c) $\quad (\partial_t^4 - \partial_t^2) w_j = \sum_{m=0}^{j-2} \alpha_m w_{j-2-m} - \sum_{m=0}^{j-1} M_{j-m} w_m, \qquad (j \geq 2)$.

We note that the equations for the boundary layer terms are ordinary differential equations. Inserting the series expansion (5.14) in the boundary conditions $u|_\Gamma = \partial_n u|_\Gamma = 0$ and noting that $\partial_n w = -\varepsilon^{-\frac{1}{2}} \partial_t w$, we find the system of coupled boundary conditions

(5.19a) $\quad v_0|_\Gamma = 0$,

(5.19b) $\quad v_j|_\Gamma + w_{j-1}|_{t=0} = 0 , \qquad j \geq 1$,

(5.19c) $\quad \left.\frac{\partial v_j}{\partial n}\right|_\Gamma - \left.\frac{\partial w_j}{\partial t}\right|_{t=0} = 0 , \qquad j \geq 1$.

These boundary conditions do not completely determine the boundary layer functions w_j, hence we add the condition that w_j is small outside the boundary layer, i.e.

(5.19d) $\quad \lim_{t\to\infty} \frac{\partial^m w_j}{\partial t^m} = 0, \qquad \text{for} \quad m = 0,1,2$.

We remark that a solution v_j of (5.17) cannot satisfy more than one boundary condition, hence the natural choice is to select for it the condition of lowest order. Any other choice will not lead to series expansions for u and λ in ascending powers of ε.

The set of equations (5.17-18-19) can be solved recursively, as we shall show. Obviously the first equation to be solved is (5.17a-19a), hence the principal terms of u and λ are an eigenfunction and an eigenvalue of the limit operator L_0. Let α_0 be an eigenvalue of L_0, let E be the corresponding eigenspace (dim E $< \infty$) and let $v_0 \in E$ be an eigenfunction of unit length, i.e. $\|v_0\| = 1$. Next we can solve w_0 from (5.18a) and the boundary conditions (5.19c-d),

(5.20) $\quad w_0(t,\cdot) = -(\partial_n v_0|_\Gamma) e^{-t}$.

Now we have to solve v_1 from (5.17b) and (5.19b). Since we prefer to solve an equation with homogeneous boundary conditions instead of (5.17b–19b), we choose a function $z_1 \in C^\infty(\Omega)$, such that

$$(5.21) \qquad z_{1|\Gamma} = \partial_n v_{0|\Gamma} ,$$

and we solve $\tilde{v}_1 := v_1 - z_1$ from

$$(5.22) \qquad \Delta\tilde{v}_1 + \alpha_0\tilde{v}_1 = -\Delta z_1 - \alpha_0 z - \alpha_1 v_0, \qquad \tilde{v}_{1|\Gamma} = 0 .$$

It is well-known from the Fredholm alternative that this equation (5.22) has a unique solution in the orthogonal complement of E provided the right-hand side $\Delta z_1 + \alpha_0 z_1 + \alpha_1 z_0$ is orthogonal to the eigenspace E. If dim E = 1 this determines the coefficient α_1 uniquely,

$$(5.23) \qquad \alpha_1(v_0,v_0) + (\Delta z_1 + \alpha_0 z_1, v_0) = 0 ,$$

and hence by Green's formula we find

$$(5.24) \qquad \alpha_1 = -(\Delta z_1 + \alpha_0 z_1, v_0)/\| v_0\|^2 = \int_\Gamma z_1(\partial_n v_0)ds/\| v_0\|^2 =$$
$$= \| \partial_n v_0\|^2_{L^2(\Gamma)}/\| v_0\|^2 .$$

We see that α_1 does not depend on the choice of z_1 and that it does not change if v_0 is multiplied by a constant; it depends on the choice of α_0 solely. If dim E = ℓ > 1, the orthogonality yields the condition

$$(5.25) \qquad \alpha_1(v_0,V) - (\partial_n v_0, \partial_n V)_{L^2(\Gamma)} = 0, \qquad \forall_{V\in E} .$$

Clearly this is a selfadjoint (non-degenerate) eigenvalue problem in E with ℓ (non-zero) eigenvalues, $\alpha_1^1,\ldots,\alpha_1^\ell$, and a corresponding set of ℓ mutually orthogonal eigenfunctions $v_0^1,\ldots,v_0^\ell$. This imposes a splitting of the eigenspace E_k according to the eigenspaces of (5.25). Our recursive procedure cannot start with an arbitrary element $v_0 \in E$; we have to choose the principal term v_0 of the expansion (5.14) within a common eigenspace of L_0 and of (5.25). With such a choice the coefficient α_1 is determined uniquely; the part $v_0 + \varepsilon^{\frac{1}{2}}v_1$ is determined uniquely if the dimension of the common eigenspace is one.

In the same way we can proceed further, we solve w_1 from (5.18b) and (5.19c-d) and we solve v_2 and α_2 from (5.17c–19b), where α_2 again is determined by an orthogonality condition of type (5.25). If the common eigenspace is still of dimension larger than one, again a splitting of it may be imposed. Thereafter we can solve w_2, v_3 and α_3 and so on, thus determining the formal power series (5.14-15) completely. Finally we cut the boundary layer terms $\sum \varepsilon^{\frac{1}{2}j} w_j$ in the expansion off by a C^∞-function χ which is zero outside the strip $0 < \rho < \rho_0$ and is one in the smaller strip $0 < \rho < \frac{1}{2}\rho_0$,

$$(5.26) \qquad \chi(\rho) = \begin{cases} 1, & \text{if } \rho < \frac{1}{2}\rho_0 , \\ 0, & \text{if } \rho > \rho_0 , \end{cases} \qquad \text{and} \qquad \chi \in C^\infty(\mathbb{R}) .$$

In this way we can construct from each $\lambda_{k,0} \in \sigma(L_0)$ the formally approximate eigenvalue Λ_k^m of order m,

$$(5.27) \qquad \Lambda_k^m(\varepsilon) = \lambda_{k,0} + \sum_{j=1}^{m} \varepsilon^{\frac{1}{2}m} \alpha_{kj} ,$$

provided the data (in casu the boundary Γ) of the problem (5.1) are sufficiently smooth. Moreover, we find the corresponding formally approximate eigenfunction U_k^m of order m,

$$(5.28) \qquad U_k^m := \sum_{j=0}^{m-1} \varepsilon^{\frac{1}{2}j} v_{kj} + \varepsilon^{\frac{1}{2}m} \psi v_{km} + \chi \sum_{j=0}^{m-2} \varepsilon^{\frac{1}{2}j+\frac{1}{2}} w_{kj} ,$$

where χ is the cut-off function (5.26) and where ψ is defined by

$$(5.29) \qquad \psi(\rho) = \{1 + \varepsilon^{\frac{1}{2}} \chi(\rho) \exp(-\varepsilon^{-\frac{1}{2}}\rho)\}/(1 + \varepsilon^{\frac{1}{2}}) .$$

The m-th order term ψv_{km} in (5.28) has to be introduced in this way, in order to ensure that the partial sum U_k^m satisfies both boundary conditions of (5.16) exactly. From the construction we find the estimates

$$(5.30) \qquad \| (L_\varepsilon - \Lambda_k^m(\varepsilon)) U_k^m \| = \mathcal{O}(\varepsilon^{\frac{1}{2}m+\frac{1}{4}}) ,$$

$$(5.31) \qquad ((L_\varepsilon - \Lambda_k^m(\varepsilon)) U_k^m, U_j^m) = \mathcal{O}(\varepsilon^{\frac{1}{2}m+\frac{1}{2}}) .$$

If $\Lambda_k^m(\varepsilon) = \Lambda_{k+1}^m(\varepsilon)$, i.e. if the multiplicity is larger than one, we can do the construction such that U_k^m and U_{k+1}^m are orthogonal (up to the order $\mathcal{O}(\varepsilon^{\frac{1}{2}m+\frac{1}{2}})$). If $\lambda_{k,0} = \lambda_{k+1,0}$ and $\Lambda_k^m \neq \Lambda_{k+1}^m$, we can order $\{\Lambda_k^m\}$ such that $\Lambda_{k+1}^m(\varepsilon) \geq \Lambda_k^m(\varepsilon)$ for all sufficiently small values of ε.

5.3. *Convergence of the eigenvalues.* Using the set of formally approximate eigenfunctions constructed above, we can derive from the minimax formula (5.7) an upper estimate for each eigenvalue. Let $k \in \mathbb{N}$ be such that $\lambda_{k+1,0} > \lambda_{k,0}$ (equality is excluded explicitly) and let V_k be the linear hull (span) of the first k formally approximate eigenfunctions,

$$V_k := \operatorname{span}\{U_j^m \mid j \in \mathbb{N},\ j \le k\} \qquad \text{(with } m \ge 1) .$$

The minimax characterization (5.7) of $\lambda_k(\varepsilon)$ now implies

$$(5.32) \qquad \lambda_k(\varepsilon) \le \max_{u \in V_k, \|u\|=1} (L_\varepsilon u, u) .$$

By (5.31) we can estimate this maximum by

$$(5.33) \qquad \lambda_k(\varepsilon) \le (L_\varepsilon U_k^m, U_k^m)/\| U_k^m \|^2 + O(\varepsilon^{\frac{1}{2}m+\frac{1}{2}}) \le \Lambda_k^m(\varepsilon) + O(\varepsilon^{\frac{1}{2}m+\frac{1}{2}}) .$$

In conjunction with the lower bound (5.12) this shows

$$(5.34) \qquad \lambda_k(\varepsilon) = \lambda_{k,0} + O(\sqrt{\varepsilon}), \qquad \varepsilon \to +0 .$$

This shows that all eigenvalues of L_ε converge to an eigenvalue of L_0 and that the interval $(-\infty, \frac{1}{2}\lambda_{k,0} + \frac{1}{2}\lambda_{k+1,0})$ contains at most k eigenvalues of L_ε, i.e. that the eigenvalues of L_ε are well-separated. This property is the key to improve the asymptotic formula (5.34) by the estimate (5.30).

Let $k \in \mathbb{N}$ be such that $\lambda_{k+1,0} > \lambda_{k,0}$ and define τ by

$$\tau := \lambda_{k+1,0} - \lambda_{k,0}, \qquad \text{(note: } \tau > 0 \text{ by the choice of k)} .$$

According to formula (5.33) an $\varepsilon_0 > 0$ exist such that

$$(5.35) \qquad \Lambda_k^m(\varepsilon) \le \lambda_{k,0} + \tfrac{1}{4}\tau \qquad \text{and} \qquad \lambda_{k+1}(\varepsilon) \ge \lambda_{k,0} + \tfrac{3}{4}\tau ,$$

for all $\varepsilon \le \varepsilon_0$. Let now $\{u_i \mid i \in \mathbb{N}\}$ be the orthonormal set of eigenfunctions of L_ε, i.e.

$$(5.36) \qquad L_\varepsilon u_i = \lambda_k(\varepsilon) u_i, \qquad (u_i, u_j) = \delta_{ij} \quad \text{(Kronecker delta)} .$$

Let us now expand the approximate eigenfunction U_j^m in the true eigenfunctions of L_ε,

$$(5.37) \qquad U_j^m = \sum_{i=1}^{\infty} (U_j^m, u_i) u_i, \qquad \| U_j^m \|^2 = \sum_{i=1}^{\infty} (U_j^m, u_i)^2 .$$

Formulae (5.30-35) imply existence of a constant C_1 such that for all $j \le k$

$$\sum_{i=k+1}^{\infty} (U_j^m, u_i)^2 = \sum_{i=k+1}^{\infty} \frac{((L_\varepsilon - \Lambda_j^m) U_j^m, u_i)^2}{(\lambda_i(\varepsilon) - \Lambda_j^m(\varepsilon))^2} \le C_1 \varepsilon^{m+\frac{1}{2}} .$$

Hence the projection P_k onto the span S_k of the first k true eigenvectors of L_ε,

$$P_k u := \sum_{i=1}^{k} (u,u_i), \qquad S_k := \mathrm{span}\{u_1,\dots,u_k\} ,$$

satisfies for every $j \le k$ the estimates

$$\text{(5.36a)} \qquad \| P_k U_j^m \| \ge (1 - C_1 \varepsilon^{m+\frac{1}{2}}) \| U_j^m \|,$$

$$\text{(5.36b)} \qquad \| (1 - P_k) U_j^m \| \le C_1 \varepsilon^{\frac{1}{2}m+\frac{1}{4}} \| U_j^m \| .$$

If $\Lambda_i^m = \Lambda_j^m$ $(i < j \le k)$, then we can choose U_i^m and U_j^m such that they are orthogonal; if $\Lambda_i^m \neq \Lambda_j^m$, then an $\varepsilon_0 > 0$ and a $\gamma > 0$ exist such that

$$|\Lambda_j^m - \Lambda_i^m| \ge \gamma \varepsilon^{\frac{1}{2}m}, \qquad (i < j \le k), \qquad \forall_{\varepsilon \le \varepsilon_0} .$$

Hence U_i^m and U_j^m satisfy the orthogonality relation

$$|(U_i^m, U_j^m)| = \left| \frac{((L_\varepsilon - \Lambda_i^m) U_i^m, U_j^m)}{\Lambda_i^m - \Lambda_j^m} \right| = \mathcal{O}(\varepsilon^{\frac{1}{4}}), \qquad (\varepsilon \to +0) .$$

The estimates (5.36) imply that the projections $P_k U_i^m$ satisfy the same relation,

$$(P_k U_i^m, P_k U_j^m) = \mathcal{O}(\varepsilon^{\frac{1}{4}}) , \qquad (\varepsilon \to +0) .$$

This implies that the set $\{U_i^m \mid i = 1,\dots,k\}$ is an approximately orthogonal basis in R_k. Since L_ε satisfies, cf. (5.30),

$$\| (L_\varepsilon - \Lambda_j^m(\varepsilon)) P_k U_j^m \| = \mathcal{O}(\varepsilon^{\frac{1}{2}m+\frac{1}{4}}) , \qquad (\varepsilon \to +0) ,$$

the matrix representation of L_ε with respect to this basis consists of a diagonal matrix, whose entries are the approximate eigenvalues Λ_j^m, plus a perturbation matrix, whose entries are of the order $\mathcal{O}(\varepsilon^{\frac{1}{2}m+\frac{1}{4}})$. Gershgorin's theorem, cf. [18], implies that the eigenvalues of this matrix are equal to $\Lambda_j^m + \mathcal{O}(\varepsilon^{\frac{1}{2}m+\frac{1}{4}})$; this proves the theorem:

THEOREM 5. *The* k*-th eigenvalue* $\lambda_k(\varepsilon)$ *of* L_ε *satisfies the asymptotic estimate*

$$\text{(5.37)} \qquad \lambda_k(\varepsilon) = \Lambda_k^m(\varepsilon) + \mathcal{O}(\varepsilon^{\frac{1}{2}m+\frac{1}{4}}), \qquad \varepsilon \to +0 ,$$

where Λ_k^m *as defined in* (5.27).

5.4. *If* Ω *is the unit disk,* the eigenvalues and eigenfunctions of L_0 are explicitly known:

$$(5.38) \qquad \mu_{k\ell}(0) = \alpha_{k\ell}^2, \qquad e_{k\ell}(r,\varphi) = J_k(\alpha_{k\ell}r)e^{ik\varphi} ,$$

where $\alpha_{k\ell}$ is the ℓ-th zero ($\ell \in \mathbb{N}$) of the k-th Bessel function J_k ($k \in \mathbb{Z}$) and where (r,φ) are polar coordinates; all eigenvalues with $k \neq 0$ have multiplicity 2. According to theorem 5, (5.38) provides the lowest order part of the expansions of eigenvalues and eigenfunctions of L_ε. Using the scheme presented in section (5.2) we find the next term by (5.25),

$$\alpha_1 = 2\pi\alpha_{k\ell}^2(J_k'(\alpha_{k\ell}r))^2/\| J_k(\alpha_{k\ell}r)e^{ik\varphi}\|^2 = \tfrac{1}{2}\alpha_{k\ell}^2 ;$$

we remark that the orthogonality condition does not impose a splitting of the eigenspace. So we find the asymptotic formula

$$(5.39) \qquad \mu_{k\ell}(\varepsilon) = \alpha_{k\ell}^2 + \tfrac{1}{2}\varepsilon^{\frac{1}{2}}\alpha_{k\ell}^2 + O(\varepsilon) ,$$

for $\varepsilon \to +0$.

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